Math 255B Lecture 12 Notes

Daniel Raban

February 3, 2020

1 Extending Symmetric Operators to Self-Adjoint Operators

1.1 Graph of the adjoint of a symmetric operator

Suppose we have a closed, symmetric, densely defined operator $S : D(S) \to H$. We introduced the **deficiency subspaces** $D_\pm = (\mathrm{Im}(S \pm i))^\perp = \ker(S^* \mp i)$ and the graphs $\widehat{D}_\pm = G(S^*)|_{D_\pm}$.

Last time, were proving the following theorem.

Theorem 1.1. *Let S be closed, symmetric, and densely defined. Then*

$$G(S^*) = G(S) \oplus \widehat{D}_+ \oplus \widehat{D}_-,$$

where the direct sum is orthogonal.

Proof. It remains to show that if $(y, S^*y) \perp G(S), \widehat{D}_\pm$, then $y = 0$.

First, if $(y, S^*y) \perp G(S)$, then $\langle (y, S^*y), (x, Sx) \rangle = 0$ for all $x \in D(S)$. Then $\langle Sx, S^*y \rangle + \langle x, y \rangle = 0$ for all $x \in D(S)$. So $S^*y \in D(S^*)$ and $S^*(S^*y) = -y$. So $((S^*)^2 + 1)y = 0$. We get $(S^* - i)(S^* + i)y = 0$, so $(S^* + i)y \in D_+$.

If $(y, S^*y) \perp \widehat{D}_+$, then $\langle (y, S^*y), (x, ix) \rangle = 0$ for all $x \in D_+$. We get $\langle y, x \rangle + \langle S^*y, ix \rangle = 0$, so $-i \langle (S^* + i)y, x \rangle = 0$ for all $x \in D_+$. So $(S^* + i)y = 0$.

Similarly, $(S^* - i)y \in D_-$ (changing the order in the factorization). Then $(y, S^*y) \perp \widehat{D}_-$, so $(S^* - i)y = 0$. So we get $y = 0$. $\square$

1.2 Conditions for extending symmetric operators

Corollary 1.1. *A symmetric, closed operator $S : D(S) \to H$ is self-adjoint if and only if the deficiency indices $n_+ = n_- = 0$, or equivalently, $\mathrm{Im}(S \pm i) = H$. Equivalently, the Cayley transform of S is unitary : $H \to H$.*

In general, we have the following:

Corollary 1.2. *A symmetric, closed operator $S : D(S) \to H$ has a self-adjoint extension if and only if the Cayley transform T can be extended to a unitary map: $H \to H$.*

Proof. This follows from our correspondence between symmetric operators and their Cayley transforms. $\square$

Using the full strength of this result we have proven, we get the original result of von Neumann's extension theory.

Theorem 1.2 (von Neumann)**.** *A closed, densely defined, symmetric operator $S : H \to H$ has a self-adjoint extension if and only if the deficiency indices are equal.*

Proof. Assume first that T can be extended to a unitary map $U : H \to H$ (so $U|_{\mathrm{Im}(S-i)} = T$). Write $H = D(T) \oplus D_-$ and $H = \mathrm{Im}(T) \oplus D_+$ (orthogonal decompositions). It follows that $U|_{D^-} : D_- \to D_+$ is a bijection, so the deficiency indices are equal: $n_- = n_+$.

Conversely, assume that $n_- = n_+$. Let $(e_j^+)_{j\in J}, (e_j^-)_{j\in J}$ be orthonormal bases for D_+ and D_-, respectively. Let $T_1 : D_- \to D_+$ take $\sum_{j\in J} x_j e_j^- \mapsto \sum_{j\in J} x_j e_j^+$. T_1 is unitary, so the map $U : H \to H$ sending $(y + z) \mapsto Ty + T_1 z$ (where $y \in D(T), z \in D_-$) is a unitary extension of T. $\square$

Remark 1.1. We say that closed, symmetric operator S is **maximal** if it has no strict symmetric extension. If S is self-adjoint, it is maximal. In general, S is maximal if and only if at least one of the deficiency indices equals 0.

1.3 Example: Extending the Schrödinger operator

Example 1.1. The Schrödinger operator: $H = L^2(\mathbb{R}^n)$, and $P = -\Delta + V(x)$, where $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R})$. Equipped with the domain $C_0^\infty(\mathbb{R}^n)$, P becomes symmetric and densely defined.

We claim that P has a self-adjoint extension. We have to check that $n_+ = \dim\ker(P^* - i) = \dim\ker(P^* + i) = n_-$. Here, $D(P^*) = \{u \in L^2 : Pu \in L^2\}$, where Pu is taken in the sense of distributions; $Vu \in L^1_{\mathrm{loc}}$, so it makes sense as a distribution. The complex conjugation map $\Gamma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ sending $u \mapsto \overline{u}$ satisfies: $\Gamma(D(P^*)) \subseteq D(P^*)$ and $[\Gamma, P^*] = 0$. Since $\Gamma : D_- \to D_+$ is a bijection, the deficiency indices are equal. (Here, we use that P is real.)